Int J Theor Phys (2010) 49: 468–496
DOI 10.1007/s10773-009-0227-3

On a Ströbel-Inspired $k(t)$ FLRW Ansatz in a Class of Metric $F(R)$ Models

M. Zouhar

Received: 5 August 2009 / Accepted: 15 December 2009 / Published online: 24 December 2009

Abstract Assuming a $D \geq 4$ dimensional FLRW (Friedmann–Lemaître–Robertson–Walker) inspired ansatz with spatial curvature being a non-trivial function of time $k(t)$ in a class of metric and Jordan frame $F(R)$ gravity models, non-existence theorems for several types of sources are derived in a simple manner (using specific form of the modified gravity Einstein tensor components).

Keywords Modified gravity · $F(R)$ · Metric formulation · Jordan frame · Inhomogeneous · Cosmology

1 Introduction and Motivation

Present day cosmology faces several challenges. One of the newest is that modern astrophysical observations [1–3] suggest that the Universe has entered (late time) accelerated expansion phase. (Note that there are several doubts concerning that conclusion [4, 5] and alternatives based on averaging not exactly homogeneous and isotropic space [6].) It seems a peculiar substance called dark energy or modification of General Relativity (GR) and perhaps both are required to model this. A review and further references are presented e.g. in [7]. Some other yet unexplained puzzles [8, 9] have co-sparked an intensive study of modified gravity models, a trend which is followed also in this article.

Some of the reasons to study the inhomogeneous $k(t)$ FLRW metric (17) are

– Astrophysical applications come to mind. Consider initial (resp. late) time constant k, it may represent an emerging (resp. decaying) spherically symmetric object in a non-perturbative regime.
– Since matter distribution in space is not perfectly homogeneous, the FLRW metric is only an approximation in an averaged model. Effective approximation of these imperfections can be taken to be an *effective* $k(t)$ FLRW metric—as can be found in [10, 11].

M. Zouhar (✉)
Department of Physical Electronics, Faculty of Science, Masaryk University, Kotlářská 2, Brno, Czech Republic
e-mail: sybok@physics.muni.cz

- Some reasons to study inhomogeneous models in general are presented in an extensive review [12] and they are demonstrated by the results of e.g. [13, 14].
- Certain modified gravity models suggest that (total) space-time is higher dimensional and $4D$ we experience can be represented as an extended object, a brane, in the total space-time. One may also approach the value of space-time dimension from a different point of view—study a general D and find if $D = 4$ is special in some sense, thus justifying one time-like and three space-like dimensions [15]. This is a motivation to study higher dimensional problems and their solutions as well.
- If k changes sign, then spatial topology is changed—a process that can be thought of in an extra-dimensions and brane-world models.

  Indeed, considering a non-trivial copy (non-constant k) with one spatial dimension, change of k from non-positive to positive values represents compactification of that dimension.

  Moreover, the above mentioned copy as a 2-dimensional (including the time) part of $N > N^* = 1$ metric is conformally flat and coordinate transformation to the conformally flat form transforms (17) into

$$ds^2 = e^{2a(t,\tilde{r})}dt^2 + e^{2a(t,\tilde{r})}d\tilde{r}^2 + \sum_{A:\text{trivial}} e^{2a_A(t,\tilde{r})}dl_A^2$$

  which reassembles some of the brane-world (inspired) ansätze studied in literature [16, 17].
- Quite recently, an article [18]—and a related comment [19]—operating with FLRW-like metric with curvature parameter k varying in time has appeared. The authors speculate of a possible stringy quantum gravity motivation for such an ansatz, which—if truly justified—adds more appeal to the metric ansatz (17).

In author's opinion, all the above items represent a stimulation for a search of an exact $k(t)$ FLRW solution to find out if it can contribute to these topics. If the answer is positive, then to what extent and under what conditions?

It is said that every metric is a solution to (modified) Einstein equations with appropriate matter part contribution(s). Of course, the matter source has to be reasonable. Author's search for an exact $k(t)$ FLRW metric (beyond Ströbel's solution [20]) has yielded some negative results and they are presented in this article.

To author's best knowledge, the problem is not studied in literature, at least in the simple manner applied (and to the extent as) in this article. Reference [12] mentions existence of a perfect fluid with heat flow standard GR solution [20] and its later re-discoveries. The GR perfect fluid (-like sources) non-existence result is rather known, see the claim in [19] referring to [21].

When compared to the above references, this work also deals with some other matter sources in $F(R)$ gravity models.

$F(R)$ model [22–28] represents one of the generalised gravity model. Some specific functions $F(R)$, e.g. $F = R + \lambda R^n$ and $F = R + \lambda R^{-1}$, were able to model accelerated expansion but they do not meet requirements that come from Solar system measurements and some other cosmological desiderata, see [29] for the former model. Attempts to find $F(R)$ models that evade the above mentioned problem lead to several candidates for the function $F(R)$ that are presented e.g. in [23–28]. One of them involves ratio of polynomials, which will also be treated—the model (28). The other model (29) seems to be promising in the standard FLRW background in the light of the dynamical system study presented in [30].

This work relates to [18] by showing it is uneasy to obtain solution of the form assumed in [18] (beyond the Ströbel's one) assuming some standard matter sources in a class of metric $F(R)$ models.

The approach adopted in this article relies on simple form of the $k(t)$ FLRW metric and consequently its (metric) Riemann tensor components in the chosen frame. Metric compatibility of the $F(R)$ gravity models is supported by [31, 32] in which well-formulation of the problem, i.e. metric compatible $F(R)$ models in presence of matter, is derived in four dimensional space-time. The paper's results on Palatini $F(R)$ are inconclusive, see recent critics [33], i.e. it is possible results in case of metric $F(R)$ do not apply to its Palatini counterpart. Then, the metric $F(R)$ may seem to be favoured at the light of the above mentioned analysis.[1]

The author believes that simplicity of this approach, though restrictive, enables for a plain judgement of admissibility of some given matter sources without the need of solving partial differential equations (that may appear in case of a more general ansatz).

This paper is written, as far as the structure concerns, as a mathematical text because the author believes it enables for a well arranged presentation which is organised as follows:

Properties of some standard matter sources in an assumed background geometry (17) are examined in Sect. 2. We obtain gravity model independent conditions—consistency relations (CRs)—quadratic in the metric stress-energy-momentum tensor of the matter sources.

Consequently a suitable mathematical framework is developed, in order to treat the above mentioned CRs effectively, in Sect. 3.

Curvature related tensors are calculated in Sect. 4. Then, specific metric and Jordan frame $F(R)$ gravity models are introduced in Sect. 5 and examination of the matter sources conditions follows in Sect. 6. The overall results are summarised in Theorem 1 (Table 2) in Sect. 7.

Units in which all $c = 8\pi G = \hbar = 1$ are used. The convention regarding space-time indices is

$$x = (x^\mu) = (x^0, (x^i)), \quad (x^i) = (x^1 = r, x^\Gamma \equiv \vartheta^\Gamma).$$

2 Matter Sources in an Assumed Background

We shall examine several familiar relativistic sources in the $k(t)$ FLRW space-time. The sources are assumed to be minimally coupled to gravity, i.e. the total Lagrangian can be separated in the matter and gravity part. Then variational equations of motion (EOM) for the metric, EOMg, can be written—in the units chosen—as

$$\text{EOM}g: \ {}^M\mathrm{G} = T, \quad T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta\sqrt{|g|}L_{\text{matter}}}{\delta g_{\mu\nu}}.$$

The ${}^M\mathrm{G}$ corresponds to the variation of the gravitational part of the Lagrangian. Since we shall deal with modified gravity models, we refer it as the generalised Einstein tensor. Its left subscript M indicates inclusion of a possible departure from the standard GR Einstein tensor into ${}^M\mathrm{G}$.

[1] Though it may be hard to obtain a general result—in the spirit of [31, 32]—concerning the Cauchy problem, examination of a perfect fluid matter source in metric affine $F(R)$ gravity shows the problem is both well-posed and well-formulated [34] thus indicating that metric $F(R)$ need not be favoured that much.

2.1 Consistency Relations (CRs)

Definition 1 Let us write a conformally flat and (spatially) spherically symmetric metric ansatz for a D-dimensional space-time ($D \geq 4$)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2a(t)}h^2\mathrm{d}l^2, \quad h^{-1} \equiv 1 + \frac{1}{4}k(t)r^2, \tag{1}$$

$\mathrm{d}l^2$ denotes a Euclidean distance in $(D-1)$-dimensions and r represents radial coordinate. The metric (1) with $\dot{k} \neq 0$, dot stands for derivative with respect to time, is called a non-trivial $k(t)$ FLRW metric.

Remark 1 This metric ansatz is closely related to the Stephani universe, see [12]. Our choice of coordinate system is different from the one widely used papers examining the aforementioned class of space-times; it makes obvious the reason of inhomogeneity lies in promoting k to a function of time.

Postulate 1 *All considered matter sources are postulated to respect the space-time symmetry. The implications for scalars ϕ and vectors v and the Yang–Mills field* (*to be denoted by A*) *strength tensor* ($F = \mathrm{D}A$), *the latter being subject to a less severe constraint*, *are presented in* (2).

$$\left.\begin{aligned} &\phi = \phi(t, r), \\ &v = v^0(t,r)\partial_0 + v^1(t,r)\partial_1, \\ &F = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \hat{F}_{\alpha\beta}(t,r)e^\alpha \wedge e^\beta, \end{aligned}\right\} \tag{2}$$

where an over-hat indicates the corresponding tensor components are written in an orthonormal frame

$$e^\alpha = e^\alpha_\mu \mathrm{d}x^\mu = \sqrt{|g_{\mu\mu}|}\delta^\alpha_\mu \mathrm{d}x^\mu.$$

We shall treat the $k(t)$ FLRW metric (1) at first and then comment on some generalisations.

Definition 2 A perfect fluid-like source(s' class) is defined by the stress-energy-momentum tensor

$$T_{\mu\nu} = \chi v_\mu v_\nu + \kappa g_{\mu\nu}, \tag{3}$$

χ and κ are scalar functionals of the matter fields, v is a vector.

Remark 2 The source class in Definition 2 implicitly contains added prescribed varying cosmological constant (shifting $\kappa \to \kappa + \Lambda(t,r)$). Moreover, if the starting Lagrangian, unspecified yet, is independent of derivative of metric, we may consider prescribed varying gravitational constant $G(t,r)$ since it can be absorbed into the parameters χ and κ by rescaling them. Such rescaling will not alter subsequent considerations.

Some representatives of the source class in Definition 2 are Bulk viscous fluid, see e.g. [35], and scalar field ϕ. The latter with Lagrangian

$$L = \ell(X, \phi) + w(\phi)\Box\phi, \quad X = \frac{1}{2}(\partial\phi)^2, \ \Box = \nabla^\rho\nabla_\rho,$$

where ℓ and w are arbitrary functions.

Both representatives are sources of interest in cosmology—indeed, scalar fields with non-canonical Lagrangian are used as dark matter models [7, 36] in a similar way to the exotic equation of state barotropic fluids, e.g. (extensions of) Chaplygin gas [37, 38].

Corollary 1 *The perfect-fluid-like source metric stress-energy-momentum tensor allows for obtaining a relation quadratic in its orthonormal frame components with no other objects entering.*

Proof The metric stress-energy-momentum tensor is

$$\left.\begin{aligned} \hat{T}_{00} &= \chi(\hat{v}_0)^2 + \kappa, \\ \hat{T}_{11} &= \chi(\hat{v}_1)^2 - \kappa, \\ \hat{T}_{\Gamma\Gamma} &= -\kappa, \\ \hat{T}_{01} &= \chi\hat{v}_0\hat{v}_1. \end{aligned}\right\} \tag{4}$$

Substituting for κ into $\hat{T}_{00}$ and $\hat{T}_{11}$ enables us to express components of the vector v and input them into $\hat{T}_{01}$ to end up with

$$0 = -\left(\hat{T}_{01}\right)^2 + \left(\hat{T}_{00} + \hat{T}_{\Gamma\Gamma}\right)\left(\hat{T}_{11} - \hat{T}_{\Gamma\Gamma}\right). \tag{5}$$

□

Corollary 2 *In case of both co-moving bulk viscous fluid with heat flow and co-moving bulk viscous fluid with null fluid, the sources' metric stress-energy-momentum tensors allow for obtaining a relation linear in their orthonormal frame components with no other objects entering.*

Proof Let us write, as in [12],

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{BVF}} + T^{\mu\nu}_{\mathrm{NF}} + T^{\mu\nu}_{\mathrm{HF}} + \Lambda g^{\mu\nu}. \tag{6}$$

The new parts of the metric stress-energy-momentum tensors are

$$T^{\mu\nu}_{\mathrm{HF}} = 2q^{(\mu}u^{\nu)}, \qquad T^{\mu\nu}_{\mathrm{NF}} = \tau k^\mu k^\nu.$$

The appearing quantities (the bulk viscous fluid velocity u, heat flow q and null fluid velocity n) are assumed to satisfy some constraints

$$1 = g(u,u) = g(k,u); \qquad 0 = g(k,k) = g(q,u) \tag{7}$$

in order to make the splitting unique.

We omit (shear-)viscous fluid and thermo-dynamical relations associated with the heat flow contribution and the EM field when compared to [12]. Since this work reports of non-existence results, based solely on EOMg, these relations are of no importance here.

Let us note that the heat flow addition can be expected due to inhomogeneity which indicates possible non-zero temperature gradient.

The metric stress-energy-momentum tensor of the source (6) has (at most) four independent orthonormal frame components which are

$$\hat{T}_{00} = (\rho + p)\hat{u}_0^2 - p + 2\hat{q}_0\hat{u}_0 + \tau\hat{k}_0^2 + \Lambda,$$
$$\hat{T}_{11} = (\rho + p)\hat{u}_1^2 + p + 2\hat{q}_1\hat{u}_1 + \tau\hat{k}_1^2 - \Lambda,$$
$$\hat{T}_{\Gamma\Gamma} = +p - \Lambda,$$
$$\hat{T}_{01} = (\rho + p)\hat{u}_0\hat{u}_1 + \hat{q}_0\hat{u}_1 + \hat{u}_0\hat{q}_1 + \tau\hat{k}_0\hat{k}_1.$$

We shall view the above relations, together with the (7), as a set of algebraical equations which can be solved with respect to the parameters that appear—namely the scalars and vectors (each of u, q and k has at most two non-zero components as implied by the space-time symmetry).

The assumption of co-moving bulk viscous fluid simplifies the metric stress-energy-momentum tensor since (7) yields

$$\hat{q}_0 = \hat{u}_1 = 0, \qquad \hat{u}_0 = \hat{k}_0 = |\hat{k}_1| = 1$$

and the metric stress-energy-momentum tensor becomes

$$\hat{T}_{00} = \rho + \tau + \Lambda,$$
$$\hat{T}_{11} = p + \tau - \Lambda,$$
$$\hat{T}_{\Gamma\Gamma} = p - \Lambda,$$
$$\hat{T}_{01} = \hat{q}_1 + s_k\tau,$$

where s_k is defined as $s_k \equiv \hat{k}_1/\hat{k}_0$. The above relations reveal

$$\tau = \hat{T}_{11} - \hat{T}_{\Gamma\Gamma}, \qquad \hat{q}_1 = \hat{T}_{01} - s_k\left(\hat{T}_{11} - \hat{T}_{\Gamma\Gamma}\right). \tag{8}$$

Assuming combination co-mov. bulk viscous fluid with heat flow imposes $\tau = 0$. Similarly assuming combination co-mov. bulk viscous fluid with null fluid imposes $\hat{q}_1 = 0$. The two equations resulting from (8) are the CRs. □

Corollary 3 *The non-linear Abelian Yang–Mills field of the form* (2) *has diagonal metric stress-energy-momentum tensor.*

Proof The non-linear Abelian Yang–Mills field is described by the Lagrangian $L = L(F^2)$ and

$$T_{\mu\nu} = -4L_F F_{\mu\rho}F_\nu^\rho + Lg_{\mu\nu}, \tag{9}$$
$$(\mathrm{EOM}A)^\mu = -2(L_F F^{\mu\rho})_{;\rho} \tag{10}$$

and the notation is $F \equiv \mathrm{D}A = \mathrm{d}A$, F^2 is traced over Yang–Mills Lie algebra indices and $L_F = \frac{\partial L}{\partial F^2}$. Non-linear EM field is an appealing generalisation of the linear case because the non-linearity can cure singularities in the field strength F, e.g. [39].

In order to obtain a non-zero T_{01} component, we must have

$$0 \neq \hat{T}_{01} \propto \hat{F}_{0\beta}\hat{F}_1^\beta = \sum_\Gamma \eta^{\Gamma\Gamma}\hat{F}_{0\Gamma}\hat{F}_{1\Gamma}.$$

Thus there should exist at least one index Γ such that the both F-components appearing in the above equation are non-vanishing. Such terms contradict the obvious space-time isotropy.

It is an easy exercise to show that operating with both EOMA and the identity $\mathrm{d}F = 0$ reveals off-diagonal terms in T cannot be produced. □

Remark 3 Trace of (9) is

$$\mathrm{Tr}\, T = DL - 4L_F F^2$$

and it vanishes in case of linear $D = 4$ Yang–Mills field. $\mathrm{Tr}\, T = 0$ can be easily extended to a special case of non-linear Yang–Mills fields with

$$L = b\left(F^2\right)^{D/4} \tag{11}$$

where b is a constant but it can also be an externally prescribed, i.e. non-dynamical, function.

Corollary 4 *Postulate 1 implies that Proca (vector) field A, with cosmological constant, matter source metric stress-energy-momentum tensor allows for obtaining a relation quadratic in its orthonormal frame components with no other objects, apart from the cosmological constant, entering.*

Proof Proca field matter source with cosmological constant is depicted by $L = \frac{1}{2}[bF^2 + m^2A^2] + \Lambda$ and

$$T_{\mu\nu} = -2bF_{\mu\rho}F^{\rho}_{\nu} - m^2 A_\mu A_\nu + L g_{\mu\nu}. \tag{12}$$

It can be viewed e.g. as the (linear) EM field with mass m.

Proca field can also be seen as a particular case of the Einstein–Aether Lorentz-breaking vector field, though normalisation condition was not imposed. Indeed, we may rearrange the Lagrangian so that it yields

$$L = \frac{1}{2}\left[bF^2 + \bar{m}^2 A^2\right] + \lambda(A^2 - 1) + \bar{\Lambda},$$

where the new parameters are

$$\bar{m}^2 = m^2 - 2\lambda, \qquad \bar{\Lambda} = \Lambda + \lambda.$$

Promoting λ to a Lagrange multiplier enforces the normalisation constraint as EOMλ. The form of EOMg is unaltered by such a rearrangement. The vector field A is time-like and it defines a preferred time-like direction, hence Lorentz violation occurs an can be studied within covariant framework [40].

The Postulate 1 restricts the vector A so that

$$\begin{aligned}
-2\hat{T}_{00} &= m^2\hat{A}_0^2 + m^2\hat{A}_1^2 + bF^2 - 2\Lambda,\\
-2\hat{T}_{11} &= m^2\hat{A}_0^2 + m^2\hat{A}_1^2 - bF^2 + 2\Lambda,\\
-2\hat{T}_{\Gamma\Gamma} &= m^2\hat{A}_0^2 - m^2\hat{A}_1^2 + bF^2 + 2\Lambda,\\
-\hat{T}_{01} &= m^2\hat{A}_0\hat{A}_1.
\end{aligned}$$

Similarly to the case of the source (3), we may eliminate the bF^2 term from both $\hat{T}_{00}$ and $\hat{T}_{11}$ by means of $\hat{T}_{\Gamma\Gamma}$ and use the $\hat{T}_{01}$ to arrive at

$$0 = -\left(\hat{T}_{01}\right)^2 + \left(\hat{T}_{00} - \hat{T}_{\Gamma\Gamma} - 2\Lambda\right)\left(\hat{T}_{11} + \hat{T}_{\Gamma\Gamma} + 2\Lambda\right). \tag{13}$$

Letting both b and m be non-dynamical functions does not affect the elimination. □

Remark 4 Vector gauge fields and vector-like fields, including massive and non-linear ones, were examined in the cosmological context e.g. in [41–44].

Corollary 5 *Metric stress-energy-momentum tensor of charged mass-less Dirac field* (*which may correspond to neutrinos when neglecting their mass*) ψ *has vanishing trace.*

Proof The Dirac (bi-)spinor field ψ is described by [45]

$$L = \frac{1}{2}\bar{\psi}[\mathrm{i}\nabla\!\!\!/ - m]\psi + \text{h.c.}$$

and the variational metric stress-energy-momentum tensor and EOMψ

$$T^{\mu\nu} = -\mathrm{i}\bar{\psi}\gamma^{(\mu}\nabla^{\nu)}\psi + \text{h.c.}, \tag{14}$$

$$\text{EOM}\psi = [\mathrm{i}\nabla\!\!\!/ - m]\psi, \tag{15}$$

where slash notation is $B\!\!\!/ \equiv \gamma^\mu B_\mu$, h.c. denotes hermitian conjugate of the preceding term(s), ∇ is gauge-covariant derivative and m is mass of the field.

Dirac spinor Lagrangian can be also written as

$$L = \frac{1}{2}\bar{\psi}\,\text{EOM}\psi + \text{h.c.},$$

i.e. it vanishes on-shell (EOMψ satisfied). This is the reason why the stress-energy-momentum tensor contains no Lg term as usual.

As a result,

$$\operatorname{Tr} T = -\mathrm{i}\bar{\psi}\nabla\!\!\!/\psi + \text{h.c.} = -2(L + m\bar{\psi}\psi) = -2m\bar{\psi}\psi.$$

Setting $m = 0$ gives the statement of the above Corollary. □

Remark 5 In retrospect, we may say five distinct (classes of) matter sources were treated.

- Perfect-fluid-like source. Two examples are scalar field and bulk viscous fluid.
- Co-moving bulk viscous fluid with either heat flow or null fluid.
- Sources with diagonal metric stress-energy-momentum tensor. Two examples are Abelian Yang–Mills field subject to Postulate 1 and co-moving bulk viscous fluid (and their non-interacting combinations).
- Sources with trace-free metric stress-energy-momentum tensor. E.g. mass-less (Yang–Mills charged) Dirac field, Yang–Mills field with the Lagrangian (11), the bulk viscous fluid with both heat flow and null fluid with equation of state $p = \frac{1}{D-1}\rho$ and their non-interacting combinations.

  The bulk viscous fluid with both heat flow and null fluid follows from (6) with $\Lambda = 0$ and the additional conditions (7) together with the chosen equation of state.
- The Proca field.

2.2 Generalising the CR

Lemma 1 *The Rastall generalisation* [46, 47] *does not alter the CRs of perfect-fluid-like sources, the co-moving bulk viscous fluid with heat flow, co-moving bulk viscous fluid with null fluid, the sources with diagonal or trace-free metric stress-energy-momentum tensor.*

Moreover, Rastall generalisation of an arbitrary matter source combination with the Rastall parameter satisfying $D\gamma = -1$ *implies* $0 = \mathrm{Tr}\, T_{\mathrm{Rastall}}$.

Proof Rastall modification adds a term proportional to metric, i.e. a diagonal contribution, by transforming the metric stress-energy-momentum tensor as follows

$$T^{\mu\nu} \to T^{\mu\nu}_{\mathrm{Rastall}} = T^{\mu\nu} + \gamma(\mathrm{Tr}\, T) g^{\mu\nu}.$$

- In case of perfect-fluid-like and co-moving bulk viscous fluid with either heat flow or null fluid models, we may view the Rastall generalisation as adding an inhomogeneous cosmological constant $\Lambda = \mathrm{Tr}\, T$. Both (5) and (8) are not altered in presence of cosmological constant.
- The diagonal metric stress-energy-momentum tensor remains diagonal, since the Rastall addition is proportional to (diagonal) metric.
- The trace of trace-free metric stress-energy-momentum tensor vanishes and so Rastall generalisation does not modify the metric stress-energy-momentum tensor at all.
- The $D\gamma = -1$ case is obvious. Trace of the Rastall generalised metric stress-energy-momentum tensor reveals

$$\mathrm{Tr}\, T_{\mathrm{Rastall}} = \mathrm{Tr}\left[T - \frac{1}{D}(\mathrm{Tr}\, T) g\right] = 0.$$

□

Lemma 2 *The Rastall generalisation of the Proca field* (12) *subject to Postulate* 1 *allows for obtaining a CR.*

Proof Taking trace of the modified EOMg enables us to express the non-Rastallized metric stress-energy-momentum tensor as

$$T_{\mu\nu} = {}^{M}G_{\mu\nu} - \frac{\gamma}{1 + D\gamma}(\mathrm{Tr}\, {}^{M}\mathrm{G}) g_{\mu\nu}.$$

We have assumed that $D\gamma \neq -1$. If $D\gamma = -1$, then one obtains $\mathrm{Tr}\, {}^{M}\mathrm{G} = 0$.

We may proceed as in the Corollary 4 to obtain

$$0 = -\left({}^{M}\hat{G}_{01}\right)^2 + \left({}^{M}\hat{G}_{00} - {}^{M}\hat{G}_{\Gamma\Gamma} - \frac{2\gamma}{1 + D\gamma}(\mathrm{Tr}\, {}^{M}\mathrm{G}) - 2\Lambda\right) \times \left({}^{M}\hat{G}_{11} + {}^{M}\hat{G}_{\Gamma\Gamma} + \frac{2\gamma}{1 + D\gamma}(\mathrm{Tr}\, {}^{M}\mathrm{G}) + 2\Lambda\right).$$

Multiplying by $[1 + D\gamma]^2$ and writing out the trace term yields

$$0 = -\left([1 + D\gamma]\, {}^{M}\hat{G}_{01}\right)^2 + \left({}^{M}\hat{Z}_{0\Gamma}\right)\left({}^{M}\hat{Z}_{1\Gamma}\right) \tag{16}$$

with the ${}^{M}\hat{Z}$-quantities defined as

$$ {}^{M}\hat{Z}_{0\Gamma} = +[1+(D-2)\gamma]\,{}^{M}\hat{G}_{00} + [2\gamma]\,{}^{M}\hat{G}_{11} - [1-(D-4)\gamma]\,{}^{M}\hat{G}_{\Gamma\Gamma} - 2[1+D\gamma]\Lambda, $$
$$ {}^{M}\hat{Z}_{1\Gamma} = +[2\gamma]\,{}^{M}\hat{G}_{00} + [1+(D-2)\gamma]\,{}^{M}\hat{G}_{11} + [1-(D-4)\gamma]\,{}^{M}\hat{G}_{\Gamma\Gamma} + 2[1+D\gamma]\Lambda. $$

One can easily see that (16) reduces to (13) in case of $\gamma = 0$. □

Definition 3 A (spatial) copy-cat of the metric (1) is defined as a spatial extension by considering more copies of spatial part of the original metric (1), i.e.

$$ \mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{A=1}^{N} \mathrm{e}^{2a_A(t)} h_A^2 \mathrm{d}l_A^2. \tag{17} $$

We may consider $\dot{k}_A = 0$ for some, but not all, copies. The copies with $\dot{k}_A \neq 0$ are called non-trivial and their total number is denoted by N^*. If the spatial dimension of a given copy A is W_A, we define $D_A \equiv W_A + 1$ (the addition $+1$ represents including the time-dimension).

The coordinates will be labelled as

$$ x = \left(x^0 = t, \left\{(x^{i_A})\right\}_{A=1}^{N}\right), $$

each copy spatial coordinates consists of a radial one and angular ones,

$$ (x^{i_A}) = (x^i_A) = \left(x^{1_A} = r_A, (x^{\Gamma_A} = \vartheta^{\Gamma_A})\right). $$

We use spherical coordinates on each copy. The "omnipresent" time coordinate $t \equiv x^0 \equiv x^0_A$ $\forall A$ is "copy-free".

We shall consider the copy-cat generalisation of the Rastall generalised CR.

Postulate 2 *All considered matter sources are postulated to respect the space-time symmetry. The implications for scalars ϕ and vectors v and the Yang–Mills field, the latter being subject to a less severe constraint, are presented in* (18).

$$ \left.\begin{aligned} \phi &= \phi(t, \{r\}^*), \\ v &= v^0(t, \{r\}^*)\partial_0 + \textstyle\sum_A^* v^{1_A}(t, \{r\}^*)\partial_{1_A}, \\ F &= F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \hat{F}_{\alpha\beta}(t, \{r\}^*) e^\alpha \wedge e^\beta, \end{aligned}\right\} \tag{18} $$

where $\{r\}^*$ is collection of radial coordinates of non-trivial copies and $\sum^*$ means sum over non-trivial copies.

Remark 6 The sources considered are extended to the copy-cat geometry by use of Postulate 2 and by the following additional prescriptions in the case of the perfect-fluid-like and co-moving bulk viscous fluid with either heat flow or null fluid sources.

Both the κ parameter and pressure p are extended to be anisotropic, i.e. we have ${}_{(A)}\kappa(t, \{r\}^*)$ and ${}_{(A)}p(t, \{r\}^*)$ where the copy-index A indicates the functions are in principle different on different copies.

Since we deal with more spatial copies, we shall consider $D_{\mathrm{copy}} = 2$ disregarded in the single-copy case ($N = 1$). It turns out this value of D_{copy} puts restrictions on fluid-like sources (namely parameters κ, implicitly including the cosmological constant Λ, and $p - \Lambda$) if we demand that CR(s) can be obtained.

Table 1 Rastall copy-cat CRs (copy index A suppressed)

Matter source, Conditions	Consistency relation (CR)
Rastall trace-free sources	$0 = \mathrm{Tr}\, T_{\mathrm{Rastall}}$
Co-mov. BVF	$0 = \hat{T}_{01}$
Co-mov. BVF + HF	$0 = \hat{T}_{11} - \hat{T}_{\Gamma\Gamma}$
Co-mov. BVF + NF, $N^* = 1$	$0 = \hat{T}_{01} - s_k(\hat{T}_{11} - \hat{T}_{\Gamma\Gamma})$
Abelian Yang–Mills, $N = 1$	$0 = \hat{T}_{01}$
Proca, $D\gamma \neq -1$, $N = 1$	Equation (16)
Perfect-fluid-like, $N = 1$	Equation (5)

BVF = Bulk Viscous Fluid, HF = Heat Flow, NF = Null Fluid

Lemma 3 *Consider free non-linear Abelian Yang–Mills field subject to Postulate 2.* EOMA *and the identity* $\mathrm{D}F = 0$ *imply that the only non-vanishing linearly independent components are/can be*

$$F_{01_A}, \quad F_{W_{A}-1:W_A}, \quad W_A \geq 3, \quad F_{1_A 1_B} \quad (B \neq A).$$

Proof Straightforward extension of the $N = N^* = 1$ (omitted) treatment. □

Lemma 4 *Assuming the same matter sources as in the single copy case, we obtain the same set of CRs if we apply following restrictions on the copy-cat space-time*

- *Co-moving bulk viscous fluid with null fluid*: $N^* = 1$, *all copies—except for one—are trivial* $k(t)$ *FLRW satisfying* $\dot{k} = 0$.
- *Perfect-fluid-like*: *only a "single" copy, i.e.* $N = 1$,
- *Proca field*: $N = 1$.
- *The Abelian Yang–Mills field*: $N = 1$.

Assume the fluid(-like) sources with $\kappa_{\mathrm{R}} = 0$ *or* $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$, *subscript* 'R' *indicates Rastall generalised quantity. The corresponding CRs retain their form in the following meaning: case of* $D_{\mathrm{copy}} > 2$ *set corresponding copy* ${}^M\hat{G}_{\Gamma\Gamma} = 0$, *an additional CR; case of* $D_{\mathrm{copy}} = 2$, *there is no* ϑ*-coordinate in this copy, hence* ${}^M\hat{G}_{\Gamma\Gamma} \equiv 0$, *no additional CR.*

The results are presented in the Table 1, copy-index is suppressed. Some details concerning the matter sources are presented in the Remark 5.

Proof Let us check each of the matter sources separately.

The $(2 \leq)\ D_{\mathrm{copy}} < 4$ will be considered in case of the fluid(-like) models since these will allow to extend the above derived CR to the copy-cat space-time with $N \geq 2$, see also Remark 6.

- Perfect-fluid-like: We have to generalise the metric stress-energy-momentum tensor in (4). v is a vector and hence it has the form given in (18), Postulate 2. This however does not suffice—then all the angular components $\hat{T}_{\Gamma_A \Gamma_A}$, and consequently corresponding components of the generalised Einstein tensor, are equal for all copies. A sufficient modification is to generalise the κg term as follows $\kappa g \to \Pi = (\Pi_{\mu\nu})$; the components are given as

$$\Pi_{00} = {}_{(0)}\kappa g_{00}, \qquad \Pi_{i_A j_B} = {}_{(A)}\kappa\, g_{i_A j_B}.$$

Then, (4) changes by appropriately labelling κ by corresponding copy-index. Thus (5) cannot be obtained unless a copy examined, say A, satisfies ${}_{(A)}\kappa = {}_{(0)}\kappa$.

In case of $N > 1$, this extension is not democratic with respect to remaining copies. Still, one may try to justify it in case of $N > N^* = 1$ by claiming that the $\dot{k} \neq 0$ copy is special and hence some sort of non-democracy can be expected. But such a reasoning is weak and we rather restrict ourselves to the case of $N = 1$.

A more detailed investigation in the geometry (17) $N \geq N^* \geq 2$ shows CRs can be obtained by using $\hat{T}_{1_A 1_B}$, $B \neq A$, components but we shall not deal with them here.

– Co-moving bulk viscous fluid with heat flow is not altered except for adding a copy-index if the pressure term is generalised as follows

$$p[g_{\mu\nu} - u_\mu u_\nu] \to \Pi_{\mu\nu}, \qquad \Pi_{00} = 0, \qquad \Pi_{i_A j_B} = {}_{(A)}p g_{i_A j_B}.$$

$\hat{T}_{\Gamma\Gamma}$ component must be involved which requires $D_{\text{copy}} > 2$. In case of $[p_{\text{R}} - \Lambda_{\text{R}}] = 0$, one obtains a CR

$$D_{\text{copy}} \geq 2: \quad 0 = -[p_{\text{R}} - \Lambda_{\text{R}}] = \hat{T}_{11}.$$

We assume the Rastall generalisation term is included within Λ_{R}.

– Co-moving bulk viscous fluid with null fluid: We shall assume the same pressure term generalisation as in case of co-moving bulk viscous fluid with heat flow. $|\hat{k}_1| = 1$ was used when deriving the CR. This need not hold since (possible ${}^M E_{1_A 1_B} \neq 0$ suggests) $\hat{k}_{1_A}$ need not vanish for more copies.

Thus proceeding as in the case of single copy yields

$$-\hat{T}_{01_A} + s_k\left(\hat{T}_{1_A 1_A} - \hat{T}_{\Gamma_A \Gamma_A}\right) = -\hat{q}_{1_A} + \tau\left(\hat{k}_{1_A}\left[-s_{k(A)} + \hat{k}_{1_A}\right]\right).$$

But $\hat{k}_{1_A}$ need not equal $s_{k(A)}$. A simple example is

$$\hat{k}_0 = 1, \qquad \hat{k}_{1_1} = \cos^2\varphi, \qquad \hat{k}_{1_2} = \sin^2\varphi,$$

with all other $\hat{k}_{1_A}$ vanishing; φ may be a function of t and $\{r\}^*$. $\hat{k}_0 = 1$ is still fixed by $g(u, k) = 1$ and by the fact that the bulk viscous fluid is co-moving.

In order to obtain a CR as in case of $N = 1$, we need $|\hat{k}_1| = 1$. This implies all other spatial sections/copies must be homogeneous and isotropic so that they do not contribute into the vector k, then only one radial component is non-vanishing and the equation $g(k, k) = 0$ yields the desired relation for the component.

The $D_{\text{copy}} = 2$ CR cannot be obtained unless $[p_{\text{R}} - \Lambda_{\text{R}}] = 0$ which implies

$$D_{\text{copy}} = 2: \quad 0 = -\hat{T}_{01} + s_k \hat{T}_{11}.$$

– Rastall trace-free metric stress-energy-momentum tensor sources: Since we used no assumptions on the space-time in the derivation of the CR, $0 = \operatorname{Tr} T$ is preserved.

– Proca field: In the single copy case, it was possible to write the metric stress-energy-momentum tensor in terms of three parameters—(two non-zero) components of the vector A and the scalar bF^2. We were able to do this because

$$2b\hat{F}_{\alpha\gamma}\hat{F}^{\gamma}_{\beta} = \left[\delta^0_\alpha \delta^0_\beta - \delta^1_\alpha \delta^1_\beta\right] bF^2, \qquad F^2 = -2(\hat{F}_{01})^2.$$

This is no longer true since

$$2b\hat{F}_{\alpha\gamma}\hat{F}^{\gamma}_{\beta} = -\sum_{B}\delta^0_\alpha\delta^0_\beta(\hat{F}_{01_B})^2 - \sum_{A,B}2\delta^0_{(\alpha}\delta^{1_A}_{\beta)}\hat{F}_{01_B}\hat{F}_{1_A1_B}$$
$$+\sum_{A,B}\delta^{1_A}_\alpha\delta^{1_B}_\beta\left(\hat{F}_{1_A0}\hat{F}_{1_B0} - \sum_{C}\hat{F}_{1_A1_C}\hat{F}_{1_B1_C}\right),$$
$$F^2 = -2\sum_{A}(\hat{F}_{01_A})^2 + 2\sum_{A>B}(\hat{F}_{1_A1_B})^2.$$

This time, writing out components involving copy A reveals for the difference and sum

$$\hat{T}_{00} - \hat{T}_{\Gamma\Gamma} = +b\left[-2\hat{F}_{0\gamma}\hat{F}^{\gamma}_0 + F^2\right] + m^2\left[-\hat{A}^2_0 + A^2\right] + 2\Lambda,$$
$$\hat{T}_{11} + \hat{T}_{\Gamma\Gamma} = +b\left[-2\hat{F}_{1\gamma}\hat{F}^{\gamma}_1 - F^2\right] + m^2\left[-\hat{A}^2_1 - A^2\right] - 2\Lambda$$

and individual components

$$\hat{T}_{01} = -2b\sum_{B\neq A}\hat{F}_{01_B}\hat{F}^{1_B}_1 - m^2\hat{A}_0\hat{A}_1,$$
$$\hat{T}_{11_B} = -2b\sum_{B\neq A}\left[\hat{F}_{10}\hat{F}_{1_B0} - \sum_{C\neq A,B}\hat{F}_{11_C}\hat{F}_{1_B1_C}\right] - m^2\hat{A}_1\hat{A}_{1_B}.$$

The copy index A was suppressed and the Rastall parameter γ does not appear because it is assumed the Rastall addition was already transfered to the gravity part of EOMg, i.e. ^{M}G.

An additional term in $\hat{T}_{01_A}$ appears so that the elimination via square of $\hat{T}_{01_A}$ is not possible.

It is clear, without explicitly writing out the A^2 and bF^2 terms, that the Proca CR (16) cannot be obtained in a general case.

Consider a specific case when all the additional terms vanish, i.e.

$$F_{01_B} = F_{1_B1_C} = 0, \quad \forall B \neq C,\ \forall C.$$

What space-time does correspond to such a situation?

Zeroing F_{01_B}, $B \neq A$ requires all other copies being both isotropic and homogeneous (A_0 is constant on these copies and $A_{i_B} = 0$, $\forall i_B$). The above condition suffices also for vanishing of $F_{1_B1_C}$, $B \neq C$.

Thence we may conclude that the additional terms do not appear (and obtaining CR is possible) in case of the $N^* = 1$ copy-cat ansatz.

Consider a case of trivial copies added. The metric stress-energy-momentum tensor components on these copies consist of a single term, namely Lg. But its orthonormal frame components are the same on all the trivial copies (and also on the non-trivial copy, the $\hat{T}_{\Gamma\Gamma}$ components).

We have no additional parameters, such as κ and p in case of the perfect-fluid-like and co-moving bulk viscous fluid with either heat flow or null fluid, to ensure the anisotropy required.

But this lack of diversity of the metric stress-energy-momentum tensor orthonormal frame components implies the same for the generalised Einstein tensor via EOMg. Such implication is unacceptable as can be seen from the standard GR model.

$$0 = \hat{G}_{yy} - \hat{G}_{\Gamma\Gamma} \quad \Rightarrow \quad \dot{k} = 0,$$

where $\hat{G}_{yy}$ stands for a trivial copy component. The implication indicated follows from Lemma 5 and it is highly undesirable. For this reason, we restrict ourselves to the case of a "single copy", $N = 1$.

Once again, restriction on the dimension of the non-trivial copy in question occurs. Equation (16) involves $\hat{T}_{\Gamma\Gamma}$ and hence it follows $D_{\text{copy}} > 2$; indeed, one cannot obtain CR in case of $D_{\text{copy}} = 2$.

– The diagonal metric stress-energy-momentum tensor sources:

The Abelian Yang–Mills field. According to Lemma 3, the metric stress-energy-momentum tensor need not remain diagonal (within each copy), the off-diagonal term is produced as follows

$$\hat{T}_{01_A} = -4L_F \sum_{B \neq A} \hat{F}_{01_B} \hat{F}^{1_B}_{1_A}.$$

Thus we restrict ourselves to $N = 1$, no copy-cat.

Co-moving bulk viscous fluid still remains co-moving and therefore diagonal. □

Remark 7 The Rastall generalised EOMg with $D\gamma = -1$ can be written also as a set of two equations

$$\left.\begin{array}{l} {}^{M}G_{\mu\nu} - T_{\mu\nu} - \frac{1}{D}(\mathrm{Tr}[{}^{M}\mathrm{G} - T])g_{\mu\nu} = 0, \\ \mathrm{Tr}\ {}^{M}\mathrm{G} = 0. \end{array}\right\} \tag{19}$$

The choice of the specific value of the Rastall parameter may seem quite peculiar but the first formula in (19) resembles the EOMg of unimodular gravity [48] with the added equation assuring the trace of the generalised Einstein tensor vanishes.

In that gravitational model, $|g|$ is non-dynamical and it therefore follows for arbitrary metric functional $\mathcal{M}[g]$, with the use of

$$0 = \delta|g| = g^{\mu\nu}\delta g_{\mu\nu},$$

that

$$0 = \frac{\delta\mathcal{M}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} \equiv \left[\delta^{\mu}_{\rho}\delta^{\nu}_{\tau} - \frac{1}{D}g^{\mu\nu}g_{\rho\tau}\right]\frac{\delta\mathcal{M}}{\delta g_{\rho\tau}}\delta g_{\mu\nu},$$

i.e. (both sides of) EOMg derived in this model are identically trace-free.

2.3 What are the CRs?

So far, the analysis was independent of the gravity model! Only the space-time symmetries were applied to the matter sources to obtain CRs. The spatial copy-cat geometry (17) makes clear that there are unknown functions of a single coordinate, the time t. Thus each CR is a set of ordinary differential equations for the unknown functions—scale factors a's and scalar curvatures k's.

Then one may ask if it is possible to read out the ordinary differential equations in a simple way. Or if it is possible to easily find some of them that tell us $\dot{k}=0$ immediately.

It turns out that the CRs, within the gravity models to be considered, can be cast into a form polynomial in both r^2 and $h(t,r)$. The terms with the highest power of h will be the ones associated with a simple ordinary differential equation, leading to $\dot{k}=0$ in most of the cases.

3 A Suitable Class of Functions—"Q-Functions"

In this section, we construct tools necessary to deal with the highest order, in h, terms of the CRs.

Definition 4 Let us define Q-functions by a representative

$$Q_{N:M}(r) \equiv \sum_{n=0}^{N}\sum_{m=0}^{M} y_{n:m} r^{2n} h^m, \quad h^{-1} \equiv 1 + \frac{1}{4}kr^2, \tag{20}$$

– $y_{n:m}$ and k are independent of r; in case of more spatial copies, we may write a Q-function

$$Q_{N_A:M_A}(\{r\}^*) = \sum_{n=0}^{N_A}\sum_{m=0}^{M_A} y_{n:m}(\{r_B\}_{B\neq A}) r_A^{2n} h_A^m.$$

– $M \geq 0$ is referred as order of $Q_{N:M}$, $y_{n:M}$ are called leading (order terms') coefficients.
– We define modulo non-leading terms $\sim$ as

$$Q_{N:M} \sim \sum_{n=0}^{N} y_{n:M} r^{2n} h^M \equiv (\sim Q_{N:M}).$$

The copy index is suppressed in what follows.
– Function $Q_{N:M}$ with $y_{n:m} = y_{m:m}\delta_{n,m}$ is called diagonal.

Lemma 5 *Stated by/in the below expression*

$$0 = Q_{N:M}(r) \quad \forall r \quad \Rightarrow \quad 0 = \sum_{n=0}^{N}\left(-\frac{1}{4}k\right)^{N-n} y_{n:M} \tag{21}$$

where $\forall r$ *means for all admissible* r *so that* $h^{-1} \neq 0$.

Proof Equation (21) multiplied by h^{-M} takes polynomial form

$$0 = \sum_{n=0}^{N}\sum_{m=0}^{M} y_{n:m} r^{2n} h^{m-M} = P_J(r) = \sum_{j=0}^{J} p_j r^{2j}, \tag{22}$$

where the (vanishing) p-coefficients can be obtained as

$$P_J(r) = \sum_{m,n,q} y_{n:m}\left(-\frac{1}{4}k\right)^q \binom{M-m}{q} r^{2(n+q)} = \sum_{m,n,j} y_{n:m}\left(-\frac{1}{4}k\right)^{j-n}\binom{M-m}{j-n} r^{2j},$$

$$p_j = \sum_{n=0}^{j}\sum_{m=0}^{M}\left(-\frac{1}{4}k\right)^{j-n}\binom{M-m}{j-n}y_{n:m}.$$

The order J of the polynomial $P_J(x)$ is

$$J = \max\{l = M+n-m \mid p_l \neq 0\} \leq M+N.$$

We have written J in a form of (general case) inequality since its concrete value heavily relies on the coefficients $y_{n:m}$.

As a result of (22), we may write the identity

$$0 = \sum_{j=0}^{J}\left(-\frac{1}{4}k\right)^{N-j} p_j. \tag{23}$$

The binomial coefficient $\binom{n}{k} \propto \Theta(n-k+1)\Theta(k+1)$, step function $\Theta(x)$ is defined as $\Theta(x \leq 0) = 0$ and $\Theta(x > 0) = 1$. Thus the step functions in binomial coefficients take care of sums over n and j and so we may treat the summations as "independent".

Let us consider fixed n and m to obtain coefficient standing at $y_{n:m}$, i.e. we will perform sum over j. It gives a simple combinatorial identity

$$\sum_j (-1)^{j-n}\binom{M-m}{j-n} = \delta(M-m)\delta(j-n).$$

Relabelling the variables, the combinatorial identity can be written as

$$\sum_l (-1)^l\binom{M-m}{l} = (1+(-1))^{M-m} = \delta(M-m).$$

The definition of the binomial coefficients assures

$$\binom{0}{l} = \delta(l).$$

Thus the examined identity (23) contains only terms with $m = M$ and $j = n$ and it simplifies to

$$0 = \sum_{n=0}^{N}\left(-\frac{1}{4}k\right)^{N-n} y_{n:M}$$

which completes the proof of (21). □

Remark 8 The above Lemma 5 can be extended to in the two following directions

- Negative powers of h, i.e. $m < 0$. Terms with $m < 0$ (are polynomials in r^2 and therefore they) can be re-interpreted as coming from the index n, i.e. the problem is reduced to the $m \geq 0$.
- Moreover, similar calculations as in the proof show that the below implication

$$0 = Q_{N:M} = \sum_{n=0}^{N}\sum_{m=M_0}^{M} y_{n:m} r^n h^m, \quad M_0 > -\infty,\ \forall r$$

$$\Rightarrow \quad \sum_{n=0}^{N} \left(-\frac{1}{4}k\right)^{\frac{1}{2}[N-n]} y_{n:M}$$

is true.

In the upcoming analysis, terms with odd n will either not appear in the leading term or only the one leading coefficient (with odd n) will be non-zero.

Lemma 6 *The set of all functions of type* (20) *form an associative commutative ring with identity denoted by* $\mathcal{R}_Q$.

Especially, finite products and finite linear combinations (*with constant coefficients*) *of functions of type* (20) *are again functions of type* (20).

Proof Since we deal with special class of functions, and functions already form ring with the specified properties, it suffices to show that $\mathcal{R}_Q$ is closed which is straightforward. □

4 The (Copy-Cat) Riemann & Related Tensors

Corollary 6 *Using the metric connection, non-vanishing independent components of the Riemann tensor* Riem *expressed in an orthonormal frame* (*being highlighted by hat above the corresponding quantity*) *are*

$$\left.\begin{aligned} \hat{R}_{0b0b} &= +\ddot{a} + (\dot{a} - \tfrac{1}{4}\dot{k}r^2h)^2 - \tfrac{1}{4}(\ddot{k} - \tfrac{1}{4}\dot{k}^2r^2h)r^2h \sim \tfrac{1}{8}\dot{k}^2(r^2h)^2, \\ \hat{R}_{\Gamma b\Gamma b} &= -(\dot{a} - \tfrac{1}{4}\dot{k}r^2h)^2 - k\mathrm{e}^{-2a} \sim -\tfrac{1}{16}\dot{k}^2(r^2h)^2, \\ \hat{R}_{\Gamma 0\Gamma 1} &= -\tfrac{1}{2}\dot{k}rh\mathrm{e}^{-a}, \end{aligned}\right\} \tag{24}$$

in $\hat{R}_{\Gamma b\Gamma b}$, $b \neq \Gamma$ *is implicitly assumed.*

The Ricci tensor is

$$\left.\begin{aligned} \hat{R}_{00} &= -\textstyle\sum_A (D_A - 1)\,{}_{(A)}\hat{R}_{0b0b} \sim_A -\tfrac{2}{16}(D_A - 1)\dot{k}^2(r^2h)^2, \\ \hat{R}_{bb} &= +\hat{R}_{0b0b} - (D_A - 2)\hat{R}_{\Gamma b\Gamma b} \sim_A +\tfrac{1}{16}D_A\dot{k}^2(r^2h)^2, \\ \hat{R}_{01} &= -(D_A - 2)\hat{R}_{0\Gamma_A 1_A\Gamma_A} = +\tfrac{1}{2}(D_A - 2)\dot{k}rh\mathrm{e}^{-a} \end{aligned}\right\} \tag{25}$$

and the scalar curvature

$$R = +\sum_A {}_{(A)}R, \tag{26}$$

with the A-copy contribution given by

$$\begin{aligned} {}_{(A)}R \;&=\; (D_A - 1)\,{}_{(A)}\Big[-2\hat{R}_{0b0b} + (D_A - 2)\hat{R}_{\Gamma b\Gamma b}\Big] \\ &\sim_A -\frac{1}{16}(D_A - 1)(D_A + 2)\dot{k}^2(r^2h)^2 \equiv (\sim_A R). \end{aligned}$$

The copy index A was suppressed whenever possible for the sake of brevity of the formulae appearing; still, it was explicitly written in case of D_A *to avoid confusion with the total space-time dimension* D.

In case of a component of a tensor $\mathcal{T}$ *with all indices from the same spatial copy A and the component independent of* r_B, $B \neq A$, *we write e.g.* ${}_{(A)}\mathcal{T}^{\mu}_{\nu\rho}$ *instead of the full* $\mathcal{T}^{\mu_A}_{\nu_A\rho_A}$.

Proof Follows from Definition 3, equation (17), and standard expressions for the Riemann tensor with metric connection. □

Corollary 7 *All non-vanishing orthonormal frame components of covariant derivatives of the scalar curvature R up to second order, shown below, are Q-functions independent of angular coordinates ϑ and they have the following leading terms*

$$
\begin{aligned}
&R \sim -\frac{1}{16}(D_A-1)(D_A+2)\dot{k}^2(r^2h)^2,\\
&\hat{\nabla}_0 R \sim -\frac{1}{2}\dot{k}r^2h(\sim R),\\
&\nabla_1 R \sim -krh(\sim R),\\
&\hat{\nabla}_0\hat{\nabla}_0 R \sim +\frac{3}{8}\dot{k}^2(r^2h)^2(\sim R),\\
&r\nabla_0\nabla_1 R \sim +\frac{1}{2}k\dot{k}(r^2h)^2(\sim R),\\
&\hat{\nabla}_b\hat{\nabla}_b R \sim -\frac{1}{8}\dot{k}^2(r^2h)^2(\sim R),\\
&\left[\hat{\nabla}_1\hat{\nabla}_1 - \hat{\nabla}_\Gamma\hat{\nabla}_\Gamma\right]R = |g^{11}|[\partial_1 - r^{-1}(1-kr^2h)]\partial_1 R \sim -2k\mathrm{e}^{-2a}(\sim R),\\
&\Box R \sim +\frac{1}{8}(D_A+2)\dot{k}^2(r^2h)^2(\sim R),\\
&(\nabla R)^2 \sim +\frac{1}{4}\dot{k}^2(r^2h)^2(\sim R)^2,
\end{aligned}
$$

the copy-index A suppressed again.

Proof Follows from Corollary 6. Obviously, "diagonality" is not preserved. Moreover, negative powers of h appear in both $\hat{\nabla}_1\hat{\nabla}_1 R$ and $\hat{\nabla}_\Gamma\hat{\nabla}_\Gamma R$. Still, the Lemma 5 holds, see Remark 8. Thus we may proceed by treating the leading terms only. □

Thus we have seen that the leading coefficients of the Riemann tensor, its contractions and Ricci scalar derivatives (up to second order) are proportional to $\dot{k}$. In case of the standard GR, the CRs leading terms will indeed yield a simple ordinary differential equation of type "constant $\times\, \dot{k} = 0$".

5 A Class of $F(R)$ Gravities

Definition 5 So called (metric) $F(R)$ models are described by the following variational principle [23, 24] $L_{\mathrm{gr.}} = F(R[g])$ resulting into generalised Einstein tensor

$$
{}^F G_{\mu\nu} = F^{(1)} R_{\mu\nu} - \frac{1}{2} F g_{\mu\nu} + [-\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box]F^{(1)}, \tag{27}
$$

$F^{(1)}$ has the following meaning $F^{(j)} = \frac{\mathrm{d}^j F}{\mathrm{d}R^j}$.

We restrict our attention to a narrow class of gravity models that can be treated easily in the adopted approach.

– The first model is

$$F = R^w \frac{P_n(R)}{P_m(R)}, \tag{28}$$

with

$$w \in \mathbb{R}, \qquad P_n = \sum_{p=0}^{n} \alpha_p R^p, \qquad P_m = \sum_{p=0}^{m} \beta_p R^p.$$

α_n and β_m are assumed to be non-vanishing, and we are primarily interested in the case of $w \in \mathbb{Z}$.

– The second model is

$$F = P_m(R) \mathrm{e}^{\lambda P_n(R)}, \quad \lambda \neq 0,\ n \geq 1, \tag{29}$$

where λ is a constant parameter.

Remark 9 Let us comment on the $F(R)$ model (28).

– We can divide the examination into two separate sub-cases $w = 0$ and $w \in (0, 1)$.
Indeed, if $w \in \mathbb{Z}$, then R^w can be absorbed into either P_n (in case of $w \geq 0$ to obtain P_{n+w}/P_m) or P_m (in case of $w \leq 0$ to end up with $P_n/P_{m+|w|}$).
If $w \notin \mathbb{Z}$, the integer-part of w can be absorbed into the ratio of polynomials. For this reason(s), it suffices to consider $w \in [0, 1)$ only.

– It will prove useful to characterise the model by $x \equiv w + n - m$.

– We do not consider cosmological constant in the gravity Lagrangian and therefore the polynomial model (28) $w = 0$ does not contain the case $n = m$.

$$\frac{P_n}{P_{m=n}} = \underline{\alpha_m \beta_m^{-1}} + \frac{P_{\bar{n}}}{P_{m=n}}, \quad \bar{n} < n = m. \tag{30}$$

The underlined constant term that can be interpreted as cosmological constant and therefore will not be considered in the gravity part of Lagrangian.
Thus $n = m$ is equivalent to “$n =$” $\bar{n} < m$ (with cosmological constant addition to the matter sector).

6 Analysis of the CR—the $F(R)$ Models

At the light of the previous sections previews and preparations, we shall cast the CR into a Q-form, find the leading terms, apply Lemma 5, (21), to obtain a simple ordinary differential equation for the curvature scalar k and examine it.

$p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ is assumed whenever we examine a non-trivial copy with $D_{\mathrm{copy}} = 2$. otherwise, $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} \neq 0$ is implicitly assumed.

The standard GR, representing a special case of $F(R)$ gravity models, will be treated first.

6.1 Standard GR

Lemma 7 *All the CR listed in Table 1, including perfect-fluid-like source with $\kappa = 0$ and co-moving bulk viscous fluid with either heat flow or null fluid with $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ in case of some copies, in the standard GR gravity model imply $\dot{k} = 0$, except for the following*

- *Diagonal matter stress-energy-momentum tensor sources that imply $(D_A - 2)\dot{k}_A = 0, \forall A$.*
- *Co-moving bulk viscous fluid with heat flow CR in case of copies A with $D_A > 2$ are identically satisfied (thus enabling the Ströbel's solution in case of $N = N^* = 1$).*

  In case of $D_A = 2$, we obtain $\dot{k}_B = 0, \forall B \neq A$, i.e. $N^* = 1$ is enforced, but the A-copy itself cannot be examined within our approach.
- *Co-moving bulk viscous fluid with null fluid $D_A = 2$ reproduces the same situation as in case of co-moving bulk viscous fluid with heat flow.*

Proof We again suppress the copy index for the sake of brevity.

- Rastall trace-free stress-energy-momentum tensor sources

$$0 = \mathrm{Tr}\,\mathrm{G} = -\frac{1}{2}(D-2)R \sim -\frac{1}{2}(D-2)(\sim R).$$

  The above leading term relation represents a simple ordinary differential equation

$$0 = (D-2)(D_A - 1)(D_A + 2)\dot{k}^2 \quad \Rightarrow \quad \dot{k} = 0.$$

- Co-moving bulk viscous fluid with heat flow is identically satisfied for all $D_A > 2$ copies. In case of $D_A = 2$ and $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$, one obtains CR

$$0 = \hat{G}_{1_A 1_A} \sim_{B \neq A} \frac{1}{2}(\sim_B R) \quad \Rightarrow \quad \dot{k}_B = 0.$$

  This reduces the copy-ansatz to $N^* = 1$. The remaining copy A yields no information.
- Diagonal metric stress-energy-momentum tensor sources and co-moving bulk viscous fluid with null fluid yields

$$0 = \hat{G}_{01_A} = \hat{R}_{01_A} \quad \Rightarrow \quad (D_A - 2)\dot{k}_A = 0.$$

  In case of the latter matter source and the (single non-trivial) copy A with $D_A = 2$, assumption of $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ leads to $0 = \hat{G}_{1_A 1_A}$ already examined above.
- The perfect-fluid-like source ($N = 1$) simplifies to

$$0 = -\left(\hat{G}_{01}\right)^2 \quad \Rightarrow \quad (D-2)\dot{k} = 0.$$

  Since we deal with a single spatial copy space-time, $D > 2$ and $\dot{k} = 0$ follows.
- The fluid(-like) sources with $\kappa = 0$ and/or $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ yield

$$0 = \hat{G}_{\Gamma\Gamma} = \hat{G}_{bb} \sim -\frac{1}{16}[D_A^2 - 2](\dot{k}r^2 h)^2 \quad \Rightarrow \quad \dot{k} = 0.$$

- The Proca field CR becomes

$$0 = (16) \sim -\frac{1}{2^8} c_0 c_1 (\dot{k}r^2 h)^4,$$

with

$$c_0 = [D - (D-2)\gamma], \qquad c_1 = [(D+1) - 2\gamma].$$

If $c_0 c_1 = 0$, we only sketch the proof without going into details.

One may examine the were-next-to-leading terms of order $M = 3$ to obtain relations $\dot{k} = \dot{k}(a)$. Then (16) becomes a Q-function/equation of order $M = 2$ of the type

$$\begin{aligned} 0 &= w_{2:2} r^4 h^2 + w_{1:2} r^2 h^2 + w_{1:1} r^2 h + w_{0:0} \\ &= \left[w_{2:2} - \frac{k}{4} w_{1:2} \right] r^4 h^2 + [w_{1:2} + w_{1:1}] r^2 h + w_{0:0}, \end{aligned}$$

where we have used $h \equiv 1 - \frac{1}{4} k r^2 h$ in the $M = 2$ term of the first line in the above expression.

Repeated applying of the Lemma 5 yields

$$0 = w_{2:2} - \frac{k}{4} w_{1:2}, \qquad 0 = w_{1:2} + w_{1:1}, \qquad 0 = w_{0:0}.$$

Using the above relations, one finds that—in both cases of either $c_0 = 0$ or $c_1 = 0$—the equations are not consistent. Thus the $k(t)$ FLRW metric cannot be a solution to them. □

Remark 10 The co-moving bulk viscous fluid with heat flow seems to be a promising candidate as to supporting the metric ansatz (17). But using the EOM for the matter, covariant divergence of the metric stress-energy-momentum tensor, yields a not very encouraging result

$$(\operatorname{Div} T)_{1_A} \sim_B q_{1_A} (\sim_B \operatorname{Div} u) \quad \Rightarrow \quad (D_A - 2)\dot{k}_A \dot{k}_B = 0 \quad \forall B \neq A.$$

Which suggests that co-moving bulk viscous fluid with heat flow may not support the $N^* > 1$ copy-cat space-time (17).

If there is no non-trivial copy with $D_{\text{copy}} > 2$, the above equation is identically satisfied. Assume $\exists C : D_C > 2$. Then the above equation restricting $\dot{k}$ yields

$$\exists C : D_C > 2 \quad \Rightarrow \quad \dot{k}_C \dot{k}_B = 0 \quad \forall B \neq C. \tag{31}$$

There are two ways to satisfy the equation (31).

$$\begin{cases} \dot{k}_C = 0 & \begin{cases} \exists H \neq C : D_H > 2 \text{ return to (31) } \& C \to H \\ \nexists H \neq C : D_H > 2 (D_H - 2)\dot{k}_H \dot{k}_B \equiv 0 \end{cases} \\ \dot{k}_B = 0 \to N^* = 1 \end{cases}$$

Repeating the analysis, one finds that either $N^* > 1$ and all non-trivial copies have $D_{\text{copy}} = 2$ or $N^* = 1$.

Still, both cases are interesting on its own. The latter one may enable for $N > N^* = 1$ generalisation of the Ströbel solution.

6.2 The $F(R)$ model (28)—$F = R^w P_n / P_m$

We are interested in the leading term of Q-form cast CRs, hence the leading term of Q-cast generalised Einstein tensor.

Then, how to turn it into a Q-form at all? Multiply the expression in question so that all negative and non-integer powers of R and P_m in denominators are eliminated. $F^{(3)}$ will in general contribute with P_m^{-4} and R^{w-3}. Thus we see that orthonormal frame components of

$$R^{3-w} P_m^4 \, {}^F\hat{G}_{\mu\nu} = \alpha_n \beta_m^3 R^{n+3m} \, {}^F\hat{\mathcal{G}}_{\mu\nu} + \cdots \tag{32}$$

are of the Q-form. The dots in the above formula indicate terms of lower order in the scalar curvature, i.e. of lower order M.

The leading term will be produced by the first term on right hand side of (32). Moreover, we may skip the non-zero passive pre-factor at the calligraphic Einstein tensor $\mathcal{E}in$.

Indeed, vanishing of the pre-factor contradicts either the assumptions on the $F(R)$ model in question ($\alpha_n \beta_m \neq 0$) or the metric (we assume non-trivial copy and $R = 0$ implies $\dot{k} = 0$) Thus it must be non-zero and we may treat it as a passive multiplicative factor.

Hence it suffices to deal solely with the calligraphic Einstein tensor. Transition to the calligraphic form will be indicated by $\simeq$.

Lemma 8 *Leading terms of the calligraphic Einstein tensor—introduced in (32)—components are given by*

$$\begin{aligned}
&{}^F\hat{\mathcal{G}}_{\alpha\beta} \sim \frac{1}{32}\epsilon_{\alpha\beta}(\sim R)^2.\\
&\epsilon_{00} \sim +(D_A - 1)[4x(x-2) + (D_A + 2)]\dot{k}^2(r^2h)^2,\\
&\epsilon_{bb} \sim +[2D_A x - (D_A - 1)(D_A + 2) - 4x(x-1)(D_A + 2x - 3)]\dot{k}^2(r^2h)^2,\\
&\epsilon_{01} \sim +16x[(D_A - 2) - (x-1)^2 kr^2]\dot{k}\mathrm{e}^{-a}r^3h,\\
&\mathrm{Tr}\,\epsilon \sim +[(D_A - 1)(D_A + 2)(D - 2x) + 4x(x-1)(D-1)(D_A + 2x - 2)]\dot{k}^2(r^2h)^2,
\end{aligned}$$

the difference

$$\epsilon_{11} - \epsilon_{\Gamma\Gamma} \sim -32x(x-1)\left[(x-2)kr^2 - 2\right]k\mathrm{e}^{-2a}.$$

In the above formulae, $x = w + n - m$.

Proof First, one finds the highest power of R in Q-cast $F^{(i)}$, which reads

$$R^{3-w} P_m^4 F^{(i)} = \left(\prod_{j=0}^{i-1}(x+j)\right)\alpha_n \beta_m^3 R^{n+3m+3-i}.$$

Then one uses Corollary 6 to determine the leading terms of all the contributing expressions involving Ricci scalar derivatives in the explicit formula for ${}^F\mathrm{G}$. □

When compared to standard GR, we have one more parameter entering the leading term coefficient—the parameter x. If suitably fixed, it may lead to vanishing of the coefficient and consequently the leading term. Then, no information on (vanishing of) $\dot{k}$ is obtained. The were-next-to-leading terms can be examined but we shall not pursue the analysis beyond the leading term here.

Lemma 9 *Leading terms of the CRs presented in Table 1—including perfect-fluid-like source with* $\kappa = 0$ *and co-moving bulk viscous fluid with either heat flow or null fluid with*

$p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ *allowed in case of some copies—with the restrictions presented below imply* $\dot{k} = 0$.

- *Diagonal metric stress-energy-momentum tensor matter sources*: *if* $D_A = 2$, *then* $F \neq P_1(R)$.
- *Co-moving bulk viscous fluid with null fluid*: *if* $D_A = 2$, *then* $x \notin \{-1/2, 1\}$.
- *Rastall trace-free matter sources*: *both*

$$\left.\begin{aligned} &0 = +\Lambda w, \\ &0 \neq +(D_A - 1)(D_A + 2)(D - 2x) + 4x(x-1)(D-1)(D_A + 2x - 2) \end{aligned}\right\} \tag{33}$$

hold simultaneously; *the latter simplifies to* $0 \neq D + 2x$ *in case of* $N = 1$ ($D_A = D$ *and so* $D - 1 \neq 0$).
- *Perfect-fluid-like source*: $D \neq 4 - 2x$
- *Co-moving bulk viscous fluid with heat flow*: $D_A \geq 2$ *with* $F \neq P_1(R)$ (*excluding the standard GR*), *moreover if* $D_A > 2$ *then* $x \neq 3/2$ *or if* $D_A = 2$ *then* $x \notin \{-1/2, 1\}$.
- *Proca field*: *both*

$$\Lambda w = 0 \neq C_0 C_1 \tag{34}$$

required to hold simultaneously.

$$\begin{aligned} C_0 &= +[1 + (D-2)\gamma]B_{\mathrm{I}} - [1 - (D-2)\gamma]B_{\mathrm{II}}, \\ C_1 &= +2\gamma B_{\mathrm{I}} + 2[1+\gamma]B_{\mathrm{II}}, \end{aligned}$$

where the B_{I} *and* B_{II} *are defined as*

$$B_{\mathrm{I}}\dot{k}^2(r^2h)^2 := \epsilon_{00}, \qquad B_{\mathrm{II}}\dot{k}^2(r^2h)^2 := \epsilon_{bb}.$$

Proof Is quite straightforward and is illustrated on a single fully treated example of a source with diagonal metric stress-energy-momentum tensor.

- The *diagonal metric stress-energy-momentum tensor matter source* CR examination shows

$$0 \simeq {}^F\mathcal{G}_{01}r \sim \frac{1}{2}x[(D_A - 2) - (x-1)^2 kr^2]\dot{k}(r^2h)^2(\sim R)^2.$$

Applying Lemma 5 yields

$$0 = \frac{1}{8}x[-(D_A - 2) - 4(x-1)^2]k\dot{k}$$

and consequently

$$0 = [(D_A - 2) + 4(x-1)^2]\dot{k}. \tag{35}$$

The implication is true because $x = 0$ is not considered, see Remark 9. Solving the constant pre-factor with respect to D_A, one obtains

$$D_A = 2 - 4(x-1)^2 \leq 2.$$

In case of $D_A > 2$, $\dot{k} = 0$ is enforced by (35).

In case of $D_A = 2$, it follows (corresponding copy) $R_{01} \equiv 0$ and we must have $x = 1$, unless $\dot{k}_A = 0$ which we try to avoid. The restriction on x allows us to write an analogue of (30)

$$\frac{P_{n=m+1}}{P_m} = \alpha_{m+1}\beta_m^{-1} R + 2\Lambda + \frac{P_{\bar{n}}}{P_m}, \quad \bar{n} < m. \tag{36}$$

The full expression for ${}^F G_{01}$ contains at least second order derivatives of F because $R_{01} = 0$. Then only the $P_{\bar{n}}/P_m$ enters ${}^F G_{01}$, thus the x can be replaced with $\bar{x} \equiv \bar{n} - m < 0$. This leads to

$${}^F G_{01} r \simeq {}^F \mathcal{G}_{01} r = -\frac{1}{2}\bar{x}(\bar{x} - 1)^2 k\dot{k} r^4 h^2 (\sim R)^2.$$

The fact that $\bar{x} \leq -1$ enables us to conclude $\dot{k} = 0$.

Examination of the remaining matter sources follows the same pattern—one applies $\simeq$, finds the leading term, obtains corresponding ordinary differential equation of the form

$$Z(D_A, x)\dot{k} = 0,$$

where Z is a polynomial function of its arguments.

If $Z(x = 0) = 0$ and $Z(x = 1) = 0$ occurs, these cases can be examined by decomposing the ratio of P_n/P_m as done in (30) and (36).

- *Rastall trace-free matter sources* and the *Proca field*: The condition Λw either sets $w = 0$, no non-integer powers of R enter CRs, or enforces $\Lambda = 0$ so that non-integer powers of R can be eliminated (by multiplying by e.g. R^{3-w} in the linear-in-metric stress-energy-momentum tensor CRs).
- *Perfect-fluid-like source*: Vanishing of the leading term coefficient's constant pre-factor of the $\simeq$ CR implies

$$0 = x[D - 2 + 4(x - 1)^2][D + 2(x - 2)]$$

  $x = 0$ need not be considered, the first square bracket term yields $D \leq 2$ and so only the last contribution remains.
- *Co-moving bulk viscous fluid with null fluid* CR's leading term is produced by ${}^F\hat{\mathcal{G}}_{01}$, i.e. it reproduces the diagonal metric stress-energy-momentum tensor matter source. □

6.3 The $F(R)$ Model (29)—$F = P_m \exp(\lambda P_n)$

Lemma 10 *All the CR in Table 1, including perfect-fluid-like source with* $\kappa = 0$ *and co-moving bulk viscous fluid with either heat flow or null fluid with* $p_R - \Lambda_R = 0$ *in case of some copies, the Proca field subject to condition* $\Lambda = 0$, *imply* $\dot{k} = 0$.

Proof Each linear (respectively quadratic) CR can be cast into the Q-form by multiplying by $\exp(-\lambda P_n)$ (respectively $\exp(-2\lambda P_n)$); sometimes an additional condition $\Lambda = 0$ must be imposed. The leading terms are those with the highest power of the parameter λ.

- Rastall trace-free stress-energy-momentum tensor sources

$$\begin{aligned} 0 &= \mathrm{Tr}\ {}^F\mathrm{G} \times \mathrm{e}^{-\lambda P_n} \\ &= (D - 1)\underline{(\nabla R)^2} P_m \left(\lambda P_n'\right)^3 + \cdots. \end{aligned}$$

The dots indicate terms of lower order. Vanishing of the not underlined part in the first term on the right hand side in the above equation is not consistent with the assumptions on this $F(R)$ model ($D \geq 4$, $\lambda \neq 0$, $P_m \neq 0$ and $n \geq 1$). Moreover, its vanishing in case of $m > 1$ and/or $n > 1$ would imply vanishing of ($\sim R$) enforcing $\dot{k} = 0$. Thus we may surpass this pre-factor—analogously to defining ${}^F\mathcal{E}in$ part of the Q-form cast generalised Einstein tensor in the $F(R)$ model (28).

The remaining relevant part is underlined and its vanishing yields

$$0 = (\nabla R)^2 \sim (\sim \hat{\nabla}_0 R)^2 \quad \Rightarrow \quad \dot{k} = 0$$

as is clear from Corollary 6.

– The diagonal metric stress-energy-momentum tensor sources CR reads

$$\begin{aligned} 0 &= {}^F G_{01} \times r\mathrm{e}^{-\lambda P_n} \\ &= \underline{(\nabla_0 R)(\nabla_1 R) r} P_m \left(\lambda P_n'\right)^3 + \cdots \quad \Rightarrow \quad \dot{k} = 0. \end{aligned}$$

– Co-moving bulk viscous fluid with heat flow CR is

$$\begin{aligned} 0 &= \left({}^F\hat{G}_{11} - {}^F\hat{G}_{\Gamma\Gamma}\right) \times \mathrm{e}^{-\lambda P_n} \\ &= \underline{(\hat{\nabla}_1 R)^2} P_m \left(\lambda P_n'\right)^3 + \cdots \quad \Rightarrow \quad \dot{k} = 0. \end{aligned}$$

The leading term is different in case of $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ $D_{\mathrm{copy}} \geq 2$. It is produced by the metric proportional part of ${}^F\hat{G}_{11}$ and the CR is

$$0 = \underline{\left[(\nabla R)^2 - (\hat{\nabla}_1 R)^2\right]} P_m \left(\lambda P_n'\right)^3 + \cdots \quad \Rightarrow \quad \dot{k} = 0.$$

– Co-moving bulk viscous fluid with null fluid matter source CR has the form

$$\begin{aligned} 0 &= \left[{}^F\hat{G}_{01} - s_k\left({}^F\hat{G}_{11} - {}^F\hat{G}_{\Gamma\Gamma}\right)\right] \times \mathrm{e}^{-\lambda P_n} \\ &= -\underline{\left[(\hat{\nabla}_0 R) - s_k(\hat{\nabla}_1 R)\right](\hat{\nabla}_1 R)} P_m \left(\lambda P_n'\right)^3 + \cdots \quad \Rightarrow \quad \dot{k} = 0. \end{aligned}$$

The conclusion remains unchanged again if we examine $p_{\mathrm{R}} - \Lambda_{\mathrm{R}} = 0$ $D_{\mathrm{copy}} \geq 2$. The leading term is due to metric proportional part of ${}^F\hat{G}_{11}$, namely the relevant part is $(\hat{\nabla}_0 R)^2$ as in the case of co-moving bulk viscous fluid with heat flow.

– The perfect-fluid-like source CR is a bit surprising—the term proportional to $(F^{(3)})^2$ vanishes identically—and it reveals

$$\begin{aligned} 0 = \Big[&-2(\hat{\nabla}_0 R)(\hat{\nabla}_1 R)(\hat{\nabla}_0\hat{\nabla}_1 R) + (\hat{\nabla}_0 R)^2([\hat{\nabla}_1\hat{\nabla}_1 - \hat{\nabla}_\Gamma\hat{\nabla}_\Gamma]R) \\ &+ (\hat{\nabla}_1 R)^2 \; ([\hat{\nabla}_0\hat{\nabla}_0 + \hat{\nabla}_\Gamma\hat{\nabla}_\Gamma]R)\Big] F^{(3)} F^{(2)} + \cdots \end{aligned}$$

and we shall pay attention to the relevant part, i.e. the square bracket $[*]$ term.

$$0 = [*] \sim -\frac{1}{2}[1 + kr^2]k\dot{k}^2 \mathrm{e}^{-2a}(r^2 h)^2(\sim R)^3 \quad \Rightarrow \quad \dot{k} = 0$$

as follows from applying Lemma 5.

- The Proca field: The requirement $\Lambda = 0$ is necessary in order to be able to cast the CR into a Q-form by multiplying it by $\exp(-2\lambda P_n)$.

  The CR becomes

$$
\begin{aligned}
0 = &+\underline{c_0 c_1 \left[(\nabla R)^2\right]^2} \left(F^{(3)}\right)^2 + 2c_0 c_1 \left[(\hat{\nabla}_0 R)^2 (\hat{\nabla}_0 \hat{\nabla}_0 R) + (\hat{\nabla}_1 R)^2 (\hat{\nabla}_1 \hat{\nabla}_1 R)\right] F^{(3)} F^{(2)} \\
&+ \left\{ -2(c_0 + c_1)^2 \underbrace{(\hat{\nabla}_0 R)(\hat{\nabla}_1 R)(\hat{\nabla}_0 \hat{\nabla}_1 R)}_{M<10} \right. \\
&+ (c_0^2 + c_1^2)[\underbrace{(\hat{\nabla}_0 R)^2 (\hat{\nabla}_1 \hat{\nabla}_1 R)}_{M=10} + \underbrace{(\hat{\nabla}_1 R)^2 (\hat{\nabla}_0 \hat{\nabla}_0 R)}_{M<10}] \\
&\left. + (c_0 - c_1)[2(D-1) - (c_0 - c_1)] \times \underbrace{(\nabla R)^2 (\hat{\nabla}_\Gamma \hat{\nabla}_\Gamma R)}_{M=10} \right\} F^{(3)} F^{(2)} + \cdots
\end{aligned}
$$

  with $c_0 = 1 - (D-2)\gamma$ and $c_1 = -2[1+\gamma]$. Orders of the expressions in curly brackets are indicated.

  $c_0 c_1 \neq 0$ yields $\dot{k} = 0$. If, on the other hand, $c_0 c_1 = 0$, the first two terms of the above relation vanish and the relevant part of the expression is determined by the curly bracket term $\{*\}$.

$$
\begin{aligned}
c_0 = 0 : 0 = \{*\} &\sim -\frac{1}{2^5}(D-2) c_1^2 \dot{k}^4 (r^2 h)^4 (\sim R)^3, \\
c_1 = 0 : 0 = \{*\} &\sim -\frac{1}{2^4} c_0^2 \dot{k}^4 (r^2 h)^4 (\sim R)^3.
\end{aligned}
$$

  Both sub-cases imply $\dot{k} = 0$. □

7 Summary and Outlook

The results of previous section can be put together into

Theorem 1 *Considered matter sources in the metric $F(R)$ gravity models do not admit a non-trivial $k(t)$ FLRW metric with the copy-cat metric ansatz* (17) *according to the Table* 2 (*which lists sources and conditions for $\dot{k} = 0$*).

Possible generalisations and outlook for future work, within the framework adopted in this article, are

- Troublesome values of parameters leading to vanishing of the constant factor in the leading term coefficient (namely adjusting D_A, x and γ) to be examined in detail. One may expect, as in the case of standard GR, that the next-to-leading terms yield relation $\dot{k}(a)$. Then, examining terms of order M lowered once more by one will include terms due to α_{n-1} and/or β_{m-1}, i.e. more detailed information on the function $F(R)$ will be required.
- Treating less simple $F(R)$ models and also $F(R, \Box)$ models—we have seen that $\Box R$ is a Q-function and so will be both $\Box^p R$ and its second order covariant derivatives.

  Indeed, the approach allows for inclusion of more complicated scalars built from the Riemann tensor into the gravity part Lagrangian. A simple example is a general order

Table 2 On Theorem 1—conditions required so that $\dot{k} = 0$

Sources	$F(R)$ [Λ not in F]	
	$R^w P_n / P_m$	$P_m \mathrm{e}^{\lambda P_n}$
Additional details	$x \equiv w + n - m$	$\lambda n \neq 0$
Diagonal sources $N^* \geq 1$	$D_A = 2$: $F \neq P_1$	–
Co-mov. BVF + HF $N^* \geq 1$	$D_A > 2$: $x \neq \frac{3}{2}$, $F \neq P_1$ $D_A = 2$: $x \notin \{-\frac{1}{2}, 1\}$	–
Co-mov. BVF + NF $N^* = 1$	$D_A = 2$: $x \notin \{-\frac{1}{2}, 1\}$	–
Perfect-fluid-like $N = 1$	$D \neq 4$: $x \neq \frac{-(D-4)}{2}$	–
$\Lambda = \text{const.}$	$\Lambda w = 0$	$\Lambda = 0$
Trace-free sources $N^* \geq 1$	Eqn. (33) for $F \neq P_1$	–
Proca $N = 1$	Eqn. (34) for $F \neq P_1$	–

BVF = Bulk Viscous Fluid, HF = Heat Flow, NF = Null Fluid

Lovelock gravity model [49] in case of which the generalised Einstein tensor is already of the Q-form. In the Lovelock model, the leading terms can be obtained by hand in case of low order (in Riemann tensor) models, e.g. linear Gauss–Bonnet addition [50, 51]. In case of higher order model, some GR/Riemannian geometry software tools can be of help.

– Include another non-equivalent (whenever $\dot{k} \neq 0$) $k(t)$ FLRW spatial section

$$f \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \quad f^{-1} \equiv 1 - k(t) r^2,$$

dΩ denotes volume element of a unit radius sphere of appropriate dimension, into the metric (17). This spatial copy has the same symmetry as those of the copy-cat metric examined, i.e. Postulate 2 remains unchanged and hence the CRs listed in Table 1 are also unaltered. Lemma 5 can still be applied; it suffices to replace the explicitly appearing k by $-4k$.

– Generalising the copy-cat ansatz (17) to a brane-world inspired forms e.g. by allowing the scale factors a_A to depend on the extra coordinate(s). Notice a non-trivial $D_A = 2$ copy already adds a brane-world flavour to an $N > N^* = 1$ $k(t)$ FLRW ansatz.
– One may also study effective EOMg on a brane. It contains terms quadratic in the metric stress-energy-momentum tensor T [52], which at least in case of the perfect-fluid-like matter source do not change the form of T so that the CR can be obtained again. The effective EOMg also contains a bulk-Weyl term, which may have a form that enables one to cast the CR into a Q-equation.

The above suggestions involve altering the gravity/geometry set-up. Combinations of (interacting) sources and massive Dirac spinor can prove to be more interesting since the leading term(s) analysis performed indicates that non-trivial $k(t)$ FLRW (space-section) is rather not allowed unless the CR is identically satisfied.

It will be interesting to find out if at least some matter sources combinations allow for obtaining an analogue of CR, possibly after at least partially solving EOM for the matter sources. A two simple examples were already examined—the Abelian Yang–Mills field, the Postulate 1 was applied to EOMA to find metric stress-energy-momentum tensor is diagonal, and the trace-free sources (see Remark 5; the matter sources combination given can be of interest especially in $D = 4$).

The analysis performed in this article shows that copies with $D_{\text{copy}} = 2$ and standard GR co-moving bulk viscous fluid with heat flow are promising candidates in the meaning that they may support an $N \geq 2$ non-trivial $k(t)$ FLRW space-time.

But we have seen the latter case becomes restricted if one considers the EOM for the matter source, see Remark 10.

We may summarise that using a simple tool, namely the Q-functions, we have obtained non-existence results/conditions concerning the metric ansatz (17), generalising the one of Ströbel, in two classes of metric $F(R)$ models. Some sub-cases indicate the space-time (17) may be supported by (some of the) the examined sources.

This approach can be applied to the copy-cat metric containing both h-copies (as in equation (1)) and f-copies (proposed in this section) in a generalised set-up, motivated by cosmology and/or astrophysics, as already discussed above. Moreover, one may consider to apply it to a suitable sub-case of a Szekeres–Szafron $\partial\beta/\partial z = 0$ family [12], which is also of cosmological importance. Considering such a geometry, its two-dimensional xy-plane has metric similar to the ansatz (1) and one may try to extend the metric by allowing the "curvature parameter" of the xy-plane to depend on time. Therefore the Q-functions seems to be a tool viable well beyond the scope of this article.

Generalising the conditions for non-existence and analysing the yet undecided sub-cases is an open issue for future work.

References

1. Riess, A.G., et al.: Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J. **116**, 1009–1038 (1998)
2. Perlmutter, S., et al.: Measurements of Omega and Lambda from 42 high-redshift supernovae. Astrophys. J. **517**, 565–586 (1999)
3. Astier, P., et al.: The supernova legacy survey: Measurement of Omega_M, Omega_ Lambda and w from the first year data set. Astron. Astrophys. **447**, 31–48 (2006)
4. Shapiro, C., Turner, M.S.: What do we really know about cosmic acceleration? Astrophys. J. **649**, 563–569 (2006)
5. Sarkar, S.: Is the evidence for dark energy secure? Gen. Relativ. Gravit. **40**, 269–284 (2008)
6. Barrow, J.D., Tsagas, C.G.: Averaging anisotropic cosmologies. Class. Quantum Gravity **24**, 1023–1032 (2007)
7. Copeland, E.J., Sami, M., Tsujikawa, S.: Dynamics of dark energy. Int. J. Mod. Phys. D **15**, 1753–1936 (2006)
8. Combes, F.: Properties of dark matter haloes. New Astron. Rev. **46**, 755–766 (2002)
9. Anderson, J.D., et al.: Indication, from Pioneer 10/11, Galileo, and Ulysses data, of an apparent anomalous, weak, long-range acceleration. Phys. Rev. Lett. **81**, 2858–2861 (1998)
10. Buchert, T.: Dark energy from structure—a status report. Gen. Relativ. Gravit. **40**, 467–527 (2008)
11. Turner, M.S., Huterer, D.: Cosmic acceleration, dark energy and fundamental physics. J. Phys. Soc. Jpn. **76**, 111015 (2007)
12. Krasiński, A.: Inhomogeneous Cosmological Models. Cambridge University Press, Cambridge (1997)
13. Ishak, M., Richardson, J., Garred, D., Whittington, D., Nwankwo, A., Sussman, R.: Dark energy or apparent acceleration due to a relativistic cosmological model more complex than flrw? Phys. Rev. D **78**, 123531 (2008)
14. Bolejko, K.: Supernovae Ia observations in the Lemaitre–Tolman model. PMC Phys. A **2** (2008)
15. Tegmark, M.: On the dimensionality of spacetime. Class. Quantum Gravity **14**, L69 (1997)
16. Dick, R.: Brane worlds. Class. Quantum Gravity **18**, R1 (2001)
17. Pérez-Lorenzana, A.: An introduction to the brane world. Lectures

18. Mersini-Houghton, L., Wang, Y., Mukherjee, P., Kafexhiu, E.: Nontrivial geometries: Bounds on the curvature of the universe. Astropart. Phys. **29**, 167–173 (2008)
19. Rasanen, S.: Comment on “Nontrivial geometries: Bounds on the curvature of the universe”. Tech. Rep. CERN-PH-TH/2007-086, CERN (2007)
20. Strobel, H.: Interior spherically symmetric solution of the Einstein field equations with thermal flow. Wiss. Z. Fr. Schiller Univ. Jena, Math. Naturw. Reihe **17**, 195 (1968)
21. Bolejko, K.: Radiation in the process of the formation of voids. Mon. Not. R. Astron. Soc. **370**, 924–932 (2006)
22. Sotiriou, T.P., Faraoni, V., Liberati, S.: Theory of gravitation theories: a no-progress report. Int. J. Mod. Phys. D **17**, 399–423 (2008)
23. Li, B., Barrow, J.D.: The cosmology of $f(R)$ gravity in the metric variational approach. Phys. Rev. D **75**, 084010 (2007)
24. Nojiri, S., Odintsov, S.D.: Introduction to modified gravity and gravitational alternative for dark energy. Int. J. Geom. Methods Mod. Phys. **4**, 115–146 (2007)
25. Hu, W., Sawicki, I.: Models of $f(R)$ cosmic acceleration that evade solar-system tests. Phys. Rev. D **76**, 064004 (2007)
26. Nojiri, S., Odintsov, S.D.: Unifying inflation with LambdaCDM epoch in modified $f(R)$ gravity consistent with solar system tests. Phys. Lett. B **657**, 238–245 (2007)
27. Cardone, F.V., Cardenas, R.P., Nodal, Y.L.: Halting eternal acceleration with an effective negative cosmological constant. Accepted
28. Starobinsky, A.A.: Disappearing cosmological constant in $f(R)$ gravity. JETP Lett. **86**, 157–163 (2007)
29. Amendola, L., Polarski, D., Tsujikawa, S.: Power-laws $f(R)$ theories are cosmologically unacceptable. Int. J. Mod. Phys. D **16**, 1555–1561 (2007)
30. Carloni, S., Dunsby, P.K.S., Troisi, A.: Cosmological dynamics of fourth order gravity. Submitted
31. Lanahan-Tremblay, N., Faraoni, V.: The Cauchy problem of $f(R)$ gravity. Class. Quantum Gravity **24**, 5667 (2007)
32. Faraoni, V.: Reply to “A comment on “The Cauchy problem of $f(R)$ gravity” ”
33. Capozziello, S., Vignolo, S.: A comment on “The Cauchy problem of $f(R)$—gravity”. Class. Quantum Gravity **24**, 5667 (2007). arxiv:0709.4414 critical comment
34. Capozziello, S.V.S.: The Cauchy problem for metric-affine $f(R)$-gravity in the presence of perfect-fluid matter. Class. Quantum Gravity **26**, 175013 (2009)
35. Misner, Thorne, Wheeler: Gravitation. Freeman, New York (1973)
36. Bertacca, D., Matarrese, S., Pietroni, M.: Unified dark matter in scalar field cosmologies. Mod. Phys. Lett. A **22**, 2893–2907 (2007)
37. Pun, C.S.J., Gergely, L.Á., Mak, M.K., Kovács, Z., Szabó, G.M., Harko, T.: Viscous dissipative Chaplygin gas dominated homogenous and isotropic cosmological models. Phys. Rev. D **77**, 063528 (2008)
38. Jamil, M., Farooq, M.U., Rashid, M.A.: Generalized holographic dark energy model. Eur. Phys. J. C **61**, 471–476 (2009)
39. Soleng, H.H.: Charged black points in general relativity coupled to the logarithmic $U(1)$ gauge theory. Phys. Rev. D **52**, 6178–6181 (1995)
40. Jacobson, T.: Einstein-aether gravity: a status report. In: From Quantum to Emergent Gravity: Theory and Phenomenology (2007)
41. Tartaglia, N.R.A.: Vector field theories in cosmology. Phys. Rev. D **76**, 083501 (2007)
42. Kiselev, V.V.: Vector field as a quintessence partner. Class. Quantum Gravity **21**, 3323–3336 (2004)
43. Hao Wei, R.-G.C.: Interacting vector-like dark energy, the first and second cosmological coincidence problems. Phys. Rev. D **73**, 083002 (2006)
44. Boehmer, T.H.C.G.: Dark energy as a massive vector field. Eur. Phys. J. C **50**, 423–429 (2007)
45. Birrell, N.D., Davies, P.: Quantum fields in curved space-time. Novokuznetsovsky Physical Mathematical Institute (NFMI), Novokuzneck (1998)
46. Rastall, P.: Generalization of the Einstein theory. Phys. Rev. D **6**, 3357 (1972)
47. Al-Rawaf, A.: Modified GR helium nucleosynthesis. Int. J. Mod. Phys. D **14**, 1941–1945 (2005)
48. Padmanabhan, T.: Dark energy: Mystery of the millennium. AIP Conf. Proc. **861**, 179 (2006)
49. Lovelock, D.: The Einstein tensor and its generalizations. J. Math. Phys. **12**, 498 (1971)
50. Shin’ichi Nojiri, S.D.O.: Modified Gauss–Bonnet theory as gravitational alternative for dark energy. Phys. Lett. B **631**, 1–6 (2005)
51. Baojiu Li, D.F.M., Barrow, J.D.: The cosmology of modified Gauss–Bonnet gravity. Phys. Rev. D **76**, 044027 (2007)
52. Daisuke Yamauchi, M.S.: A brane world in an arbitrary number of dimensions without Z_2 symmetry. Prog. Theor. Phys. **118**, 245–256 (2007)